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Application of Accelerated Iterative Methods for Solution of Thermal Models of Spacecraft

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Lumped parameter thermal network formulation is widely used for constructing mathematical models of spacecraft. For steady-state analysis, the mathematical model is a system of coupled nonlinear algebraic equations. Generally, Newton's method is used for the solution of nonlinear systems. This is an iterative method in which a linear system is to be solved at every iteration. A direct method such as Gaussian elimination is commonly employed for this purpose. However, for large order systems, conventional direct methods are impractical because of the required high storage and computation time. The coefficient matrix associated with the mathematical model of spacecraft, in most cases, is sparse. Iterative methods offer special advantages over direct methods in solving large, sparse linear systems. They are easy to code and do not require explicit storage of the coefficient matrix. Moreover, certain acceleration schemes speed up considerably the convergence of the basic iterative methods. The Ng accelerated Jacobi method and successive overrelaxation method employing Carré's algorithm for an estimate of the optimum relaxation parameter are some of the powerful iterative techniques. Results of numerical experiments conducted on various methods indicate that accelerated iterative techniques seem to be an efficient way of solving large mathematical models.

Nomenclature

C	= conduction exchange factor matrix
f	= vector of functions, $[f_1(x) = 0, f_2(x) = 0, \ldots, f_n(x) = 0]^T$
J	= Jacobian matrix, $J_{ij} = \partial f_i / \partial x_j$
n	= number of nodes or equations, order of matrix
q_a	= rate of absorbed albedo load (reflected solar load from the Earth)
q_e	= rate of absorbed earthshine load
q_p	= internal power dissipation
q_s	= rate of absorbed solar load
R	= radiation exchange factor matrix
$S(H)$	= spectral radius of the square matrix H
T	= temperature
w	= relaxation parameter
x	= vector of unknowns $(x_1, x_2, \ldots, x_n)^T$
∂	= partial derivative operator
σ	= Stefan–Boltzmann constant
∞	= infinity
$\|\cdot\|$	= vector norm

Subscripts

opt	= optimum
∞	= maximum norm
$*$	= exact root condition

Superscripts

(k)	= iteration counter
T	= transpose
-1	= inverse
$'$	= derivative

Introduction

IN steady-state thermal analysis of spacecraft, lumped parameter network formulation is used extensively to generate mathematical models.[1] The inherent simplicity and flexibility of this method makes even difficult problems manageable. In this method, the spacecraft is first divided into a number of isothermal nodes, then an energy balance expression is written for each node. The governing steady-state equation for the ith node, in mathematical form is as follows:

$$q_{s,i} + q_{a,i} + q_{e,i} + q_{p,i} + \sum_{j=1}^{n} C_{ij}(T_j - T_i) + \sum_{j=1}^{n} \sigma R_{ij}\left(T_j^4 - T_i^4\right) = 0 \quad (1)$$

Equation (1) is written for all the nodes and yields a system of coupled nonlinear algebraic equations, the solution of which gives the steady-state temperature distribution of the spacecraft.

There are two important methods for the solution of nonlinear systems: 1) Newton's method and 2) iterative methods.[2,3] Generally, Newton's method is used. The behavior of an iterative method directly applied to a nonlinear system cannot be predicted because no such general theory exists. Newton's method applied to a nonlinear system, however, leads to the solution of a linear system. The numerical solution of linear systems is founded on a firm analytical base and is well developed.

This paper focuses mainly on the applicability of accelerated iterative techniques for solving the linear system associated with Newton's method for the solution of Eq. (1). Numerical experiments have been conducted to compare the performances of various techniques.

Newton's Method

Newton's method for the solution of a nonlinear system of the form $f(x) = 0$ in n variables is basically an iterative method and can be derived as follows. Let $x^{(k)}$ be an approximation to the root x^* at the kth iteration. A Taylor series expansion of $f(x)$ about $x^{(k)}$ gives

$$f(x) = f\left[x^{(k)}\right] + J\left[x^{(k)}\right]\left[x - x^{(k)}\right] + O\left[x - x^{(k)}\right]^2 = 0 \quad (2)$$

Assuming that $[x - x^{(k)}]$ is so small that its square and higher powers can be neglected, we get

$$f\left[x^{(k)}\right] + J\left[x^{(k)}\right]\left[x - x^{(k)}\right] = 0 \quad (3)$$

Rearranging Eq. (3) and taking x as the $(k+1)$th approximation to the root leads to the celebrated Newton's method

$$J\left[x^{(k)}\right]\left[x^{(k+1)} - x^{(k)}\right] = -f\left[x^{(k)}\right] \quad (4)$$

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Eq. (4) is used iteratively to obtain successive new approximations to x^* until convergence is reached.

The fundamental theorem concerning the convergence of Newton's method states that if in the neighborhood of x^*, f is twice continuously differentiable and the Jacobian is nonsingular, and if the starting approximation is near to x^*, then

$$\lim_{k\to\infty} x^{(k)} = x^*$$

Global convergence of Newton's method is possible if, and only if, $f'' > 0$; i.e., the function should be convex. Newton's method has a quadratic rate of convergence; i.e.,

$$\left\|x^{(k)} - x^{(k-1)}\right\| = K\left\|x^{(k-1)} - x^{(k-2)}\right\|^2, \quad \text{where } K \text{ is a constant}$$

The attractive property of a quadratic rate of convergence makes Newton's method a good choice for the solution of nonlinear systems. With starting values sufficiently close to the solution, Newton's method generally converges quickly. A direct method, like Gaussian elimination, or an iterative method, like Jacobi, Gauss–Seidel, or successive overrelaxation (SOR), may be used for the solution of a linear system such as Eq. (4).[4,5]

The major computational tasks in implementing Newton's method are 1) evaluating the Jacobian and 2) solving the accompanying linear system. There is a variant of the basic Newton method, the modified Newton method, which does not require evaluation of the Jacobian at every iteration, but rather at either the first iteration or at a particular interval of iterations. This reduces the number of Jacobian evaluations, thus reducing expense. Another advantage is that it is not necessary to factor the Jacobian matrix at every iteration. The same Jacobian in factored form may be used in the following iterations, until the next updating, for the back substitution process. This reduces the solution cost considerably because the factorization process alone takes the major part of the computation effort. In the modified Newton method, however, the quadratic convergence rate is lost, and it may take a greater number of iterations, but the resulting gain in overall computing time is often significant enough to make the method attractive.

The method of solution for Eq. (4) is a crucial factor in deciding the efficiency of Newton's method because a major part of the computation efforts is expended there. For a moderate-size system, a direct method may be suitable. However, as the size of the system increases, the application of a direct method becomes impractical because of high storage and computation time requirements. An important property of the Jacobian associated with the steady-state thermal mathematical model of spacecraft is that they exhibit randomly packed sparse structure. There are sophisticated sparse matrix direct methods for the solution of sparse linear systems.[6,7] These methods are suitable for large order systems with a special sparse structure. Iterative methods are ideal for the solution of large, randomly packed, sparse linear systems. The advantages of iterative methods are that they are easy to code and require less storage. Moreover, there are powerful acceleration schemes to speed up the convergence of the basic iterative algorithms. The following section outlines some of the basic iterative methods and acceleration schemes for the solution of linear systems.

Iterative Methods for Solution of Linear Systems

Iterative methods determine the solution based on successive approximations.[8] The iterative method to solve $\boldsymbol{Ax} = \boldsymbol{b}$, where A is a square matrix of order n, and $\boldsymbol{x}$ and $\boldsymbol{b}$ are column vectors of dimension n, may be written in general form as

$$\boldsymbol{x}^{(k+1)} = H\boldsymbol{x}^{(k)} + \boldsymbol{c} \tag{5}$$

where H is the iteration matrix based on A, and $\boldsymbol{c}$ is a column vector based on A and $\boldsymbol{b}$. H and $\boldsymbol{c}$ are defined for three different iterative methods later in this section. The necessary and sufficient condition for the convergence of the iterative scheme of Eq. (5) is that the spectral radius (i.e., the modulus of the eigenvalue of the largest modulus) of H must be less than unity. The rate of convergence is defined by $\log[1/S(H)]$, $S(H)$ being the spectral radius. Therefore, an efficient iterative technique aims for an H with small spectral radius. The Jacobi method, the Gauss–Seidel method, and SOR are examples of iterative methods.

Jacobi Method

This method uses

$$H = -D^{-1}(L+U), \qquad c = D^{-1}b \tag{6}$$

where L, D, and U are defined by $A = L + D + U$, L and U are lower and upper triangular matrices with vanishing diagonal elements, and D is the diagonal matrix.

The Jacobi method is perhaps the simplest iterative scheme. This method exhibits slow convergence characteristics, and hence it is not widely used. However, certain acceleration schemes can significantly enhance its convergence speed. Aitken's method and the Chebychev scheme usually are used to accelerate the Jacobi method.[3,8]

Auer[9] presents a powerful and inexpensive technique, originally attributed to Ng, that can be used to accelerate the speed of any linearly convergent iteration scheme. The Ng method is a vector analog of Aitken's process, and it obtains the accelerated estimate as a linear combination of successive iterates of the unaccelerated scheme as

$$y = \left(1 - \sum_{j=1}^{m} a_j\right)x^{(k)} + \sum_{j=1}^{m} a_j x^{(k-j)} \tag{7}$$

where $\boldsymbol{y}$ is the vector of accelerated estimate of x, m is the order of the method, and the vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)^T$ contains the coefficients used for acceleration. The coefficient of $x^{(k)}$ is chosen so that the sum of all coefficients must be unity; thus Eq. (7) will always give $y = x$ when $x^{(k)} = x^{(k-1)} = \cdots = x^{(k-m)} = x$. The term

$$\left(1 - \sum_{j=1}^{m} a_j\right)$$

in that equation ensures this.

The coefficient vector $\boldsymbol{a}$ is evaluated as follows. The method in Eq. (7) applied after $(k-1)$ and k iterations gives two estimates: y' and y, respectively. The vector $\boldsymbol{a}$ is found by the least square minimization of the distance between y and y'; i.e., $r^2 = (y-y')^T(y-y')$ should be minimum. This results in a linear system of normal equations as

$$Aa = b \tag{8}$$

The elements of the symmetric matrix A of order m and the vector $\boldsymbol{b}$ of dimension m are defined as the dot product of two vectors of dimension n as given below:

$$A_{ij} = \left[\Delta x^{(k)} - \Delta x^{(k-i)}\right]^T\left[\Delta x^{(k)} - \Delta x^{(k-j)}\right] \tag{9}$$

$$b_i = \left[\Delta x^{(k)} - \Delta x^{(k-i)}\right]^T \Delta x^{(k)} \tag{10}$$

where Δ is the operator defined as $\Delta x^{(k)} = x^{(k)} - x^{(k-1)}$.

Equation (8) is solved to obtain the unknown acceleration coefficient vector $\boldsymbol{a}$. The Ng method of order m requires $m+2$ successive estimates.

Auer[9] reports on the numerical performance of the Ng method in solving radiative transfer problems. He gives favorable numerical results and suggests that usually a value of $m = 2$ is sufficient for many problems. The computing necessary to perform the acceleration is insignificant relative to the cost of a single iteration; however, it can achieve a considerable reduction in the number of iterations required. The Jacobi method tailored with the Ng acceleration could be a powerful iterative technique in solving linear systems.

The implementation of an mth order Ng accelerated Jacobi iterative method might look this. After every $m+2$, Jacobi iterations form the coefficient matrix A and the right-hand side vector $\boldsymbol{b}$ using Eqs. (9) and (10), and solve the system Eq. (8) for the vector of acceleration coefficients $\boldsymbol{a}$. Then, apply Eq. (7) to get the accelerated estimate of x. The process is continued until convergence.

Gauss–Seidel Method

The Gauss–Seidel method employs

$$H = -(D+L)^{-1}U, \qquad c = (D+L)^{-1}b \tag{11}$$

This method is about twice as fast as the Jacobi method, as is evident from the fact that the spectral radius of H here is the square of the associated Jacobi H. Still, the slow convergence property of this method makes it unsuitable for practical problems.

SOR Method

This is a modification of the Gauss–Seidel method and uses

$$H = (D+wL)^{-1}[(1-w)D - wU], \qquad c = (D+wL)^{-1}b \tag{12}$$

where w is the relaxation parameter ($1 < w < 2$), which is introduced in view to reduce the spectral radius of the iteration matrix. A slight deviation in w can significantly influence the rate of convergence. The key point is to estimate the optimum w (w_{opt}) for which the spectral radius of H is minimum. If w_{opt} is used, SOR works many times faster than the basic Gauss–Seidel method. For certain linear systems, especially those resulting from the finite difference approximations of some partial differential equations, w_{opt} can be found beforehand by analytical methods.[8] However, for a general linear system, this is difficult. Young's[8] famous result for w_{opt}, for a class of matrices possessing property "A" and "consistent" ordering is

$$w_{\text{opt}} = 2\Big/\left(1 + \left\{1 - [S(H_{\text{Jacobi}})]^2\right\}^{\frac{1}{2}}\right) \tag{13}$$

This shows that w_{opt} can be calculated if the spectral radius of the associated Jacobi iteration matrix is known. Spectral radius is more difficult and expensive to estimate than the solution itself. Our aim is only to solve the system, not to find the spectral radius. Carré[10] has proposed an adaptive determination of w_{opt} when the SOR iterations are in progress. His idea is to make use of the successive difference vectors of x iterates to estimate $S(H_{\text{SOR}})$ based on a power method of determination of dominant eigenvalue. It can be easily proved that

$$x^{(k)} - x^{(k-1)} = H_{\text{SOR}}\left[x^{(k-1)} - x^{(k-2)}\right] \tag{14}$$

This shows that the successive difference vectors seem to be the iterates in the power method for finding the dominant eigenvalue of H_{SOR}. Therefore, as k approaches infinity

$$S(H_{\text{SOR}}) = \left\|x^{(k)} - x^{(k-1)}\right\| \Big/ \left\|x^{(k-1)} - x^{(k-2)}\right\| \tag{15}$$

Hence, it is seen that $S(H_{\text{SOR}})$ can be obtained as a byproduct during the course of SOR iterations. After a finite number of iterations, we get only an approximate value of $S(H_{\text{SOR}})$. Three successive values of $S(H_{\text{SOR}})$ obtained with Eq. (15) may be used with Aitken's extrapolation scheme to get a better estimate. $S(H_{\text{Jacobi}})$ can be obtained from an equation that connects the spectral radii of Jacobi and SOR iteration matrices according to

$$S(H_{\text{Jacobi}}) = [S(H_{\text{SOR}}) + w - 1]\Big/\left\{w[S(H_{\text{SOR}})]^{\frac{1}{2}}\right\} \tag{16}$$

$S(H_{\text{Jacobi}})$ is used in Eq. (13) to obtain w_{opt}.

Carré's SOR algorithm may be summarized as follows. The SOR iteration begins with a proper starting value of w, which is known to be less than w_{opt}. After every, 10 or 20 iterations, apply Eq. (15) to estimate $S(H_{\text{SOR}})$, and substitute this in Eq. (16) to obtain $S(H_{\text{Jacobi}})$; w is updated using Eq. (13). The process is repeated until w reaches the optimum value. Iteration is stopped whenever convergence occurs for x. Carré's method is an economical way to estimate w_{opt} because the computation involved is insignificant, as compared to the cost of a single iteration.

Table 1 Size of the models

Model number	Number of nodes	Number of conduction exchange factors	Number of radiation exchange factors
1	76	114	314
2	94	126	443
3	111	185	642
4	1124	2972	1111
5	1146	2796	7448

Numerical Experiments

Numerical experiments have been carried out to study the efficiency of different numerical methods for the solution of spacecraft steady-state thermal mathematical models. Fortran 77 programs have been developed for this purpose. A number of mathematical models have been solved with these programs. The same model has been solved by different methods under identical values of starting temperatures and iteration-stopping criterian to make the performance comparison meaningful. All the numerical experiments have been carried out on the mainframe computer UNIVAC-1100/70 available at the ISRO Satellite Centre.

Test Computer Programs

The following six numerical methods have been implemented in test computer programs. The method indicated after the slash mark (/) denotes the one used to solve the associated linear system.

1) Newton/Gaussian elimination.
2) Modified Newton/Gaussian elimination. (Here the Jacobi was evaluated once every two iterations.)
3) Newton/Jacobi method.
4) Newton/Jacobi method accelerated by second-order Ng method.
5) Newton/Gauss–Seidel.
6) Newton/SOR, which adaptively determines the optimum relaxation parameter based on Carré's algorithm.

Whenever iterative methods are used to solve the linear system, the sparsity of the coefficient matrix has been exploited in storage, as well as in computation. A sparse row-wise format storage scheme has been used for this purpose.[11] Convergence was assumed for Newtonian iterations when $\|x^{(k)} - x^{(k-1)}\|_\infty < 5.0E-3$. The criterion $\|x^{(k)} - x^{(k-1)}\|_\infty / \|x^{(k-1)}\|_\infty < 1.0E-6$ was used to stop iterative procedures for solving linear systems.

Test Mathematical Models

Five different thermal mathematical models of spacecraft have been considered for numerical experiments. A measure of the size of a thermal mathematical model based on a lumped mass method may be defined by the number of nodes, the number of conduction exchange factors, and the number of radiation exchange factors. Computer storage and the computation times basically are determined by the preceding numbers. The order of the Jacob matrix is the same as the number of nodes.

The fill-in of the Jacob matrix is determined by the number of exchange factors. The number of radiation exchange factors is also an indication of the amount of computation because it determines the number of third- and fourth-power calculations of temperatures required for the evaluation of the Jacob and the function Eq. (1). The size of the models are given in Table 1.

Results and Discussion

Comparisons of various methods were made with respect to the CPU times. The computer used was not a vector machine. Table 2 presents the relative CPU times taken by each numerical method to solve the five mathematical models. Relative CPU time is defined here as the ratio of the CPU time taken by each method to the CPU time taken by the method that took the minimum. It can be seen that the Newton/SOR method (method 6) is the most efficient among all in 4 out of 5 cases. The only exception was for model 3, where the modified Newton method took the least time. The Newton/SOR method took CPU times (minutes) of 1.011 and 3.224 in solving models 4 and 5, respectively.

Table 2 Relative CPU times for various methods

Serial number	Method	Relative CPU time				
		Model 1	Model 2	Model 3	Model 4	Model 5
1	Newton/Gaussian elimination	1.79	2.64	1.27	[a]	[a]
2	Modified Newton/Gaussian elimination	1.50	2.06	1.00	[a]	[a]
3	Newton/Jacobi	2.95	3.18	6.59	8.43	5.03
4	Newton/Jacobi with Ng acceleration	1.07	1.05	2.02	1.53	1.53
5	Newton/Gauss–Seidel	2.08	2.10	4.33	7.44	3.64
6	Newton/SOR with Carré's w formula	1.00	1.00	1.28	1.00	1.00

[a]Not carried out because of high storage requirement.

The number of iterations taken by an accelerated iterative scheme is considerably less than that required for the corresponding basic method to solve linear systems. Take, for example, the first Newton iteration for every model. The Ng accelerated Jacobi method was found to be respectively 5.7, 2.9, 4.2, 9.3, and 4.1 times faster than the basic Jacobi method for the five models. SOR, which used Carré's algorithm to estimate the optimum relaxation parameter, was found to be respectively 4.1, 2.2, 5.7, 12.2, and 6.9 times faster than the basic Gauss–Seidel method. These speed ratios vary in every Newton iteration because the coefficient matrix changes.

Summary

Accelerated iterative methods are found to have a potential to solve the linear system accompanied with Newton's method for the solution of large size, steady-state thermal mathematical models of spacecraft. SOR tailored with an adaptive determination of the optimum relaxation parameter and the Jacobi method accelerated by the Ng scheme are good choices because of their computational efficiency and programming simplicity.

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